Canonical Quantization of Gravity without "Frozen Formalism"

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Abstract

We write down a quantum gravity equation which generalizes the Wheeler-DeWitt one in view of including a time dependence in the wave functional. The obtained equation provides a consistent canonical quantization of the 3-geometries resulting from a "gauge-fixing" $(3+1)$-slicing of the space-time.

Our leading idea relies on a criticism to the possibility that, in a quantum space-time, the notion of a $(3+1)$-slicing formalism (underlying the Wheeler-DeWitt approach) has yet a precise physical meaning.

As solution to this problem we propose of adding to the gravity-matter action the so-called *kinematical action* (indeed in its reduced form, as implemented in the quantum regime), and then we impose the new quantum constraints.

As consequence of this revised approach, the quantization procedure of the 3-geometries takes place in a fixed reference frame and the wave functional acquires a time evolution along a one-parameter family of spatial hypersurfaces filling the space-time.

We show how the states of the new quantum dynamics can be arranged into an Hilbert space, whose associated inner product induces a conserved probability notion for the 3-geometries.

Finally, since the constraints we quantize violate the classical symmetries (i. e. the vanishing nature of the super-Hamiltonian), then a key result is to find a (non-physical) restriction on the initial wave functional phase, ensuring that general relativity outcomes when taking the appropriate classical limit. However we propose a physical interpretation of the kinematical variables which, based on the analogy with the so-called *Gaussian reference fluid*, makes allowance even for such classical symmetry violation.

1 General Statements

It is commonly believed that the canonical methods of quantization, in spite of their successful predictions on many elementary particles phenomena (it stands up the great agreement of quantum electrodynamics with experimental data), nevertheless they can not be extrapolated up to arbitrarily high energies (i. e. arbitrarily small distances), when the "granular" nature of the fields and their interaction with the underlying space-time are expected to be important.
Indeed this point of view is confirmed by the large number of renormalization procedure are required to get satisfactory predictions already on Minkowski space and moreover by the inconsistencies or ambiguities characterizing the canonical quantization of "matter" field when referred to a curved space-time [1].

The implementation of the canonical formalism to the gravitational field quantization, leads to the so-called Wheeler-DeWitt equation (WDE) [2, 3], consisting of a functional approach in which the states of the theory are represented by wave functionals taken on the 3-geometries and, in view of the requirement of general covariance, they do not possess any real time dependence.
Due to its hyperbolic nature, the WDE is characterized by a large number of unsatisfactory features [4] which strongly support the idea that is impossible any straightforward extension to the gravitational phenomena of procedures well-tested only in limited ranges of energies; however in some contexts, like the very early cosmology [6, 5] (where a suitable internal time variable is provided by the universe volume) the WDE is not a dummy theory and give interesting information about the origin of our classical universe, see [7], which may be expected to remain qualitatively valid even for the outcoming of a more consistent approach.

Over the last ten years the canonical quantum gravity found its best improvement in a reformulation of the constraints problem in terms of the so-called *Ashtekar variables*, leading to the *loop quantum gravity theory* [9, 8]; this more recent approach overcomes some of the WDE limits, like the problem of constructing an appropriate Hilbert space, but under many aspects is yet a theory in progress.

The aim of this paper, more than to be the answer to the hard question whether canonical quantum gravity has some predictive issue, concerns a criticism on a fundamental *ansatz* on which is based the whole formalism: the *ansatz* we criticize is that, in a fully covariant quantum theory has yet meaning to speak of an Arnowitt-Deser-Misner (ADM) formalism [10, 3] to perform a $(3 + 1)$-slicing of the space-time (as implicitly assumed in the WDE approach).
Indeed, the space or time character of a vector, in particular of the normal to a 3-hypersurface, is sensitive to the metric field and can not be definite precisely when the gravitational field is in a quantum state (unless we consider simply a perturbation theory).
By other words we claim that the quantization procedure does not commutes with the space-time slicing operation; therefore only two approaches appear to be acceptable and self-consistent (see also the beginning of Section 4):

i) The requirement of a covariant quantization of gravity is preserved, but any slicing of the space-time prevented,
ii) The slicing representation of the space-time is allowed, but any notion of full covariance given up.
The first point of view was addressed in [11] and find a promising development in the recent issue of the so-called *spin foam* formalism [12].
The analysis here presented addresses the second point of view as leading statement and by using the discussion of the canonical methods of quantization presented in [13], as a fundamental paradigm, we get a reformulation of the WDE which overcomes many of its shortcomings.

In [13] is provided a wide discussion about the canonical methods of quantization as referred to different systems (with a satisfactory guide to the previous literature on this subject) in order to construct a quantum geometrodynamics on the base of its similarities and differences with other theories (for a valuable and more recent review on the canonical quantum gravity and the related problem of a physical time see [14]. Of particular interest is to be regarded for our purposes the analysis concerning the quantum theory of "matter" field on a fixed background, which shows the necessity of adding the so-called *kinematical action* to achieve a satisfactory structure for the quantum constraints.

Our main statement is that the kinematical term may be retained even in the gravitational case since in the ADM action the lapse function and the shift vector can be thought as assigned (like in the "matter" field case) up to a restriction on the initial Cauchy problem, i. e. the super-Hamiltonian and super-momentum constraints have to be satisfied on the initial space-like hypersurface; indeed to fix the reference frame, i. e. the lapse function and the shift vector, is equivalent to loss the super-Hamiltonian and the super-momentum constraints which (equivalent to the $0-\mu$-components of the Einstein equations), however, have an envolutive character, i. e. if satisfied on the initial hypersurface, they are preserved by the dynamical evolution [13].
Thus from a classical point of view, we add the kinematical action to the gravity-matter one, but we preserve the general relativity framework simply by assigning a particular Cauchy problem. When passing to the quantization procedure we get equations describing the evolution of a wave functional no longer invariant under time displacements (the invariance under 3-diffeomorphisms is yet retained), but therefore characterized by a dependence on the choice of the spatial hypersurface, i. e. the "time variable".

By taking the evolution of the wave functional along a one-parameter family of spatial hypersurfaces, filling the space-time, we show how the space of the solutions for the wave equation can be turned into an Hilbert space and the quantum dynamics can be reduced to a Schrödinger-like approach with an associated eigenvalue problem; it is important to stress that the quantum evolution of the 3-geometries is, at the end, expressed directly in terms of the parameter labeling the hypersurfaces (the so-called label time).

As last but fundamental achievement we find the quantum correspondence of the restriction imposed classically on the Cauchy data to provide general relativity; indeed we show that if the phase of the initial wave functional satisfies the usual Hamilton-Jacobi equation, then the classical limit $\hbar \rightarrow 0$ always reproduces the Einsteinian dynam-

ics. This feature of the model is crucial for its viability because ensures that, though violated on quantum level, the right symmetries (i. e. the vanishing nature of the super-Hamiltonian) are restored on the classical limit.

Since the approach here presented overlap in the formalism the so-called multi-time formulation of canonical quantum gravity, as well as its smeared Schrödinger version,[15, 14], it is worth stressing differences and similarities. Other important approaches based on the same (so-called) embedding variables, and even referred to the path integral formalism, can be found in [16]-[18] (see also [19]).

In this well-known quantization scheme, the kinematical variables are identified with non-physical degrees of freedom of the gravitational field, extracted by an ADM resolution of the Hamiltonian constraints. Thus, unlike for our proposal, where the kinematical variables are added by hand (as for any other field), in the multi-time approach, no violation takes place of the classical symmetries.
The main similarity is indeed only the form of the quantum equations because the physical and dynamical meaning is made significantly different in view of the different hamiltonian densities (or smeared hamiltonians) respectively involved: in the present approach we have to do with the super-Hamiltonian, while in the multi-time formalism with the ADM reduced one.

More physical insight is obtained on the nature of the kinematical variables by making a parallel between our model and the so-called *Gaussian reference fluid* [20, 14]; we argue that the formalism below developed corresponds to a generalization of the Gaussian case to a *generic reference fluid* which can play the role of a physical clock. The physical characterization of this fluid is provided by analysing the Hamiltonian equations involving the kinematical variables; as a result we show that to this generic fluid can be associated the energy-momentum tensor of a dust.

In Section 2 we give a review of the quantum theory of the "matter" field in the spirit of the discussion presented in [13], which constitutes the appropriate line of thinking for the reformulation of the quantum geometrodynamics developed in Section 4 and following a schematic derivation of the WDE approach presented in Section 3. By section 5 we compare our proposal with the Schrödinger multi-time formalism, devoting particular attention to a minisuperspace model. In Section 6 we give a discussion on the physical nature of our time variable, based on a generalization of the Gaussian fluid clock. Finally Section 7 is devoted to brief concluding remarks and provides an application of the proposed quantum gravity theory to a very simple model.

2 Quantum Fields on Curved Background

We start by a brief review of the canonical quantization of a "matter" field on an assigned space-time background [1, 13].

Let us consider a four-dimensional pseudo-riemannian manifold $\mathcal{M}^4$, characterized by a metric tensor $g_{\mu\nu}(y^\rho)$ $(\mu, \nu, \rho = 0, 1, 2, 3)$ (having signature $-, +, , +, +$) and perform a $(3+1)$-slicing of the space-time [10, 3], by a one-parameter family of spacelike

hypersurfaces $\Sigma_t^3 : y^\rho = y^\rho(t, x^i)$ $(i = 1, 2, 3)$, each of which defined by a unique fixed value of t. The normal field to the family of hypersurfaces $n^\mu(y^\rho)$ $(n_\mu n^\mu = -1)$ and the three tangent vectors to this family $e_i^\mu \equiv \partial_i y^\mu$ (we adopted the notation $\partial_i(\ ) \equiv \partial(\ )/\partial x^i$) form a reference basis in $\mathcal{M}^4$.

Thus we may decompose the so-called deformation vector $N^\mu \equiv \partial_t y^\mu$ $(\partial_t(\ ) \equiv \partial(\ )/\partial t)$ along the basis $\{\mathbf{n}, \mathbf{e}_i\}$, i. e.

$$N^\mu \equiv \partial_t y^\mu = N n^\mu + N^i e_i^\mu \tag{1}$$

where $N(t, x^i)$ and $N^i(t, x^i)$ are respectively called the *lapse function* and the *shift vector*.
By performing on the metric tensor the coordinates transformation $y^\mu = y^\mu(t, x^i)$ and using (1) , the line element rewrites

$$ds^2 = g_{\mu\nu} dy^\mu dy^\nu = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{2}$$

where $h_{ij} \equiv g_{\mu\nu} e_i^\mu e_j^\nu$ is the metric tensor induced on the hypersurfaces Σ_t^3. By the line element (2) we get the expression of the contravariant normal vector in the new coordinates (t, x^i), i. e. $\mathbf{n} = (1/N, -N^i/N)$; by the inverse components of the metric in these same coordinates, i. e. $g^{\mu\nu} \rightarrow \{-1/N^2\ N^i/N^2\ h^{ij} - N^i N^j/N^2\}$, we get the covariant vector field $\mathbf{n} = (-N, \mathbf{0})$. These contravariant and covariant expressions for the normal field imply respectively $n^\mu e^i_\mu = N^i/N$ and $n_\mu e_i^\mu = 0$

Now, within the ADM formalism, we analyze the quantization of a self-interacting scalar field $\phi(t, x^i)$ described by a potential term $V(\phi)$ on a fixed gravitational background; its dynamics is summarized by the action

$$S^\phi(\pi_\phi, \phi) = \int_{\mathcal{M}^4} \left\{ \pi_\phi \partial_t \phi - N H^\phi - N^i H_i^\phi \right\} d^3x dt \tag{3}$$

where π_ϕ denotes the conjugate field to the scalar one and . the hamiltonian terms H^ϕ and H_i^ϕ read explicitly

$$H^\phi \equiv \frac{1}{2\sqrt{h}} {\pi_\phi}^2 + \frac{1}{2}\sqrt{h} h^{ij} \partial_i \phi \partial_j \phi + \sqrt{h} V(\phi) \quad H_i^\phi \equiv \partial_i \phi \pi_\phi \tag{4}$$

being $h \equiv det h_{ij}$. This action should be varied with respect ro π_ϕ and ϕ, but not N, N^i and h^{ij} since the metric background is assigned; However it remains to specify the geometrical meaning of the lapse function and the shift vector with respect ro the space-time slicing; this aim is reached by adding to S^ϕ the so-called kinematical action [13]

$$S^k(p_\mu, y^\mu) = \int_{\mathcal{M}^4} \{ p_\mu \partial_t y^\mu - N^\mu p_\mu \} d^3x dt \tag{5}$$

so getting, by (1) the full action for the system

$$S^{\phi k} \equiv S^\phi + S^k = \int_{\mathcal{M}^4} \left\{ \pi_\phi \partial_t \phi + p_\mu \partial_t y^\mu - N(H^\phi + H^k) - N^i(H_i^\phi + H_i^k) \right\} d^3x dt \tag{6}$$

$$H^k \equiv p_\mu n^\mu \quad H^k_i \equiv p_\mu e^\mu_i \tag{7}$$

In the above action n^μ and h^{ij} are to be regarded as assigned functionals of $y^\mu(t, x^i)$. Adding the kinematical action does not affect the field equation for the scalar field, while the variations with respect to p_μ and y^μ provide the equation (1) and the evolution of the kinematical momentum.

Finally, by varying, now even, with respect to N and N^i we get the Hamiltonian constraints

$$H^\phi = -p_\mu n^\mu \quad H^\phi_i = -p_\mu e^\mu_i \tag{8}$$

Clearly is to be assigned the following Cauchy problem assigned on a regular initial hypersurface $\Sigma^3_{t_0}$, i. e. $y^\mu(t_0, x^i) = y^\mu_0(x^i)$

$$\phi(t_0, x^i) = \phi_0(x^i) \quad \pi_\phi(t_0, x^i) = \pi_0(x^i) \quad y^\phi(t_0, x^i) = y^\mu_0(x^i) \quad p_\mu(t_0, x^i) = p_{\mu\,0}(x^i) \tag{9}$$

At last, to complete the scheme of the field equations, we have also to specify the lapse function, the shift vector and the metric tensor $h_{ij} = h_{ij}(y^\mu)$,

This system can be easily quantized in the canonical formalism by assuming the states of the system be represented by a wave functional $\Psi(y^\mu(x^i), \phi(x^i))$ and implementing the canonical variables $\{y^\mu, p_\mu, \phi, \pi_\phi\}$ to operators $\{\hat{y}^\mu,\ \hat{p}_\mu = -i\hbar\delta(\ )/\delta y^\mu,\ \hat{\phi},\ \hat{\pi}_\phi = -i\hbar\delta(\ )/\delta\phi\}$. Then the quantum dynamics is described by the equations

$$i\hbar n^\mu \frac{\delta\Psi}{\delta y^\mu} = \hat{H}^\phi\Psi = \left[-\frac{\hbar^2}{2\sqrt{h}}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi} + \frac{1}{2}\sqrt{h}h^{ij}\partial_i\phi\partial_j\phi + \sqrt{h}V(\phi)\right]\Psi \quad i\hbar e^\mu_i \frac{\delta\Psi}{\delta y^\mu} = \hat{H}^\phi_i\Psi = -i\hbar\partial_i\phi\frac{\delta\Psi}{\delta\phi} \tag{10}$$

These equations have $5\times\infty^3$ degrees of freedom, corresponding to the values taken by the four components of y^μ and the scalar field ϕ in each point of a spatial hypersurface. In (10) y^μ plays the role of "time variable", since it specifies the choice of a particular hypersurface $y^\mu = y^\mu(x^i)$.

In view of their parabolic nature, equations (10) have a space of solutions that, by an heuristic procedure, can be turned into an Hilbert space whose inner product reads

$$\langle\Psi_1 \mid \Psi_2\rangle \equiv \int_{y^\mu = y^\mu(x^i)} \Psi_1^*\Psi_2 D\phi \quad \frac{\delta\langle\Psi_1 \mid \Psi_2\rangle}{\delta y^\mu} = 0 \tag{11}$$

where Ψ_1 and Ψ_2 denote two generic solutions and $D\phi$ the Lebesgue measure defined on the ϕ-function space. The above inner product induces the conserved functional probability distribution $\varrho \equiv \langle\Psi \mid \Psi\rangle$.

The semiclassical limit of this equations (10) is obtained when taking $\hbar \to 0$ and, by setting the wave functional as

$$\Psi = expi\left\{\frac{1}{\hbar}\Sigma(y^\mu, \phi)\right\} \tag{12}$$

and then expanding Σ in powers of $\hbar/i$, i. e.

$$\Sigma = \Sigma_0 + \frac{\hbar}{i}\Sigma_1 + \left(\frac{\hbar}{i}\right)^2 \Sigma_2 + ... \tag{13}$$

By substituting (12) and (13) in equations (10), up to the zero-order approximation, we find the Hamilton-Jacobi equations

$$-n^\mu \frac{\delta \Sigma_0}{\delta y^\mu} = \frac{1}{2\sqrt{h}}\left(\frac{\delta \Sigma_0}{\delta \phi}\right)^2 + \sqrt{h}\left(\frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi + V(\phi)\right) \quad e^\mu_i \frac{\delta \Sigma_0}{\delta y^\mu} = -\partial_i\phi \frac{\delta \Sigma_0}{\delta \phi} \tag{14}$$

which lead to the identification $\Sigma_0 \equiv S^{\phi k}$.

3 The Wheeler-DeWitt Equation

Now we briefly recall how the Wheeler-DeWitt approach [2, 13] faces the problem of quantizing a coupled system consisting of gravity and a real scalar field, which implies also the metric field now be a dynamical variable. The action describing this coupled system reads

$$S^{g\phi} = \int_{\mathcal{M}^4} \left\{\pi^{ij}\partial_t h_{ij} + \pi_\phi \partial_t \phi - N(H^g + H^\phi) - N^i(H^g_i + H^\phi_i)\right\} d^3x dt \tag{15}$$

where π^{ij} denotes the conjugate momenta to the three-dimensional metric tensor h_{ij} and the gravitational super-hamiltonian H^g and super-momentum H^g_i takes the explicit form

$$H^g \equiv \frac{16\pi G}{c^3} G_{ijkl}\pi^{ij}\pi^{kl} - \frac{c^3}{16\pi G}\sqrt{h}{}^3R \quad G_{ijkl} \equiv \frac{1}{2\sqrt{h}}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_k) \tag{16}$$

$$H^g_i \equiv -2^3\nabla_j \pi^j_i \tag{17}$$

In the above expressions ${}^3R$ and ${}^3\nabla(\ )$ denote respectively the Ricci scalar and the covariant derivative constructed by the 3-metric h_{ij}, while G is the Newton constant,

Since now N and N^i are, in principle, dynamical variables, they have to be varied, so leading to the constraints $H^g + H^\phi = 0$ and $H^g_i + H^\phi_i = 0$ which are equivalent to the $\mu-0$-components of the Einstein equations and therefore play the role of constraints for the Cauchy data. It is just this restriction on the initial values problem, a peculiar difference between the previous case, at fixed background, and the present one; in fact, now, on the regular hypersurface $t = t_0$, the initial conditions $\{\phi_0(x^i),\ \pi_0(x^i),\ h_{ij\,0}(x^i),\ \pi_{ij\,0}(x^i)\}$ can not be assigned freely, but they must verify on $\Sigma^3_{t_0}$ the four relations $\{H^g + H^\phi\}_{t_0} = \{H^g_i + H^\phi_i\}_{t_0} = 0$.

Indeed behaving like Lagrange multipliers, the lapse function and the shift vector have not a real dynamics and their specification corresponds to assign a particular slicing of $\mathcal{M}^4$, i. e. a system of reference.

In order to quantize this system we assume that its states be represented by a wave functional $\Psi(\{h_{ij}\}, \phi)$ (the notation $\{h_{ij}\}$ means all the 3-geometries connected by a 3-diffeomorphism) and implement the canonical variables to operators acting on this wave functional (in particular we set $h_{ij} \rightarrow \hat{h}_{ij}$, $\pi^{ij} \rightarrow \hat{\pi}^{ij} \equiv -i\hbar\delta(\ )/\delta h_{ij}$).
The quantum dynamics of the system is then induced by imposing the operatorial translation of the classical constraints, which leads respectively to the Wheeler-DeWitt equation and to the supermomentum constraint one [2]:

$$(\hat{H}^g + \hat{H}^\phi)\Psi = 0 \quad (\hat{H}^g_i + \hat{H}^\phi_i)\Psi = 0 \tag{18}$$

which to be explicited requires a specific choice for the *normal ordering* of the operators.

Due to its hyperbolic nature this formulation of the quantum dynamics has some limiting feature [4] , which we summarize by the following four points:
i) It doesn't exist any general procedure allowing to turn the space of the solutions into an Hilbert one and so any appropriate general notion of functional probability distribution is prevented.
ii) The WDE doesn't contain any dependence on the variable t or on the function y^μ, so loosing its evolutive character along the slicing Σ^3_t. Moreover individualizing an internal variable which can play the role of "time " is an ambiguous procedure which doesn't lead to a general prescription.
iii) The natural semiclassical limit, provided by a wave functional of the form

$$\Psi(\{h_{ij}\}, \phi) = expi\left\{\frac{\Sigma(\{h_{ij}\}.\phi)}{\hbar}\right\} \tag{19}$$

leads, up to the zero order of approximation $\Sigma \equiv \Sigma_0(\{h_{ij}\}, \phi)$, to the right Hamilton-jacobi equation for the classical scalar tensor action $S^{g\phi}$, but, in the limit corresponding to a fixed background $N = N^*$ $N^i = N^{i*}$ $h_{ij} = h^*_{ij}$, the WDE doesn't provide neither the Hamilton-Jacobi equation (14) expected associated to the semiclassical limit $\Psi(\{h^*_{ij}\}, \phi) = expi\{(\Sigma_0(\phi))/\hbar\}$ $(\hbar \rightarrow 0)$, neither the appropriate quantum dynamics on a fixed background (10). in correspondence to the wave functional $\Psi = \Psi(\{h^*_{ij}\}, \phi) \equiv \chi(\phi)$, which instead provides

$$\hat{H}^\phi\chi = 0 \quad \hat{H}^\phi_i\chi = 0 \tag{20}$$

This discrepancy is due to the absence of a dependence on y^μ in the action $S^{g\phi}$, which can not be clearly restored anyway.
iv)At last we stress what is to be regarded as an intrinsic inconsistency of the approach above presented: the WDE is based on the primitive notion of space-like hypersurfaces, i. e. of a time-like normal field, which is in clear contradiction with the random behavior of a quantum metric field [21]; indeed the space or time character of a vector becomes a precise notions only in the limit of a perturbative quantum gravity theory. This remarkable ambiguity leads us to infer that there is inconsistency between the requirement of a wave equation (i. e. a wave functional) invariant, like the WDE one, under space

diffeomorphisms and time displacements on one hand, and, on the other one, the (3 + 1)-slicing representation of the global manifold.

The existence of these shortcomings in the WDE approach, induces us to search for a better reformulation of the quantization procedure which addresses the solution of the above indicated four points as prescriptions to write down new dynamical quantum constraints.

4 Reformulation of the Quantum Geometrodynamics

We start by observing that, within the framework of a functional approach, a covariant quantization of the 4-metric field is equivalent to take the wave amplitude $\Psi = \Psi(g_{\mu\nu}(x^\rho))$; in the WDE approach, by adopting the ADM slicing of the space-time, the problem is restated in terms of the following replacement

$$\Psi(g_{\mu\nu}(x^\rho)) \rightarrow \Psi(N(t, x^l), N^i(t, x^l), h_{ij}(t, x^l)) \quad . \tag{21}$$

Then, since the lapse function N and the shift vector N^i are cyclic variables, i. e. their conjugate momenta p_N and p_{N^i} vanish identically, we get, on a quantum level, the following restrictions:

$$p_N = 0, \; p_{N^i} = 0 \quad \Rightarrow \quad \frac{\delta\Psi}{\delta N} = 0, \; \frac{\delta\Psi}{\delta N^i} = 0 \quad ; \tag{22}$$

by other words, the wave functional Ψ should be independent of N and N^i. Finally, the super-momentum constraint leads to the dependence of Ψ on the 3-geometries $\{h_{ij}\}$ (instead on a single 3-metric tensor h_{ij}).

The criticism to the WDE approach, developed at the point iv) of the previous section, concerns with the ill-defined nature of the replacement (21). The content of this section is entirely devoted to reformulate the quantum geometrodynamics, by preserving the (3 + 1)-representation of the space-time, but avoiding the ambiguity above outlined in the WDE approach.

Since, as well known [13], the Hamiltonian constraints $H^g + H^\phi = 0$ and $H^g_i + H^\phi_i = 0$, once satisfied on the initial hypersurface, are preserved by the remaining Hamilton equations, then the above variational principle (15) would be equivalent to a new one with assigned lapse function and shift vector (so loosing the Hamiltonian constraint), but with the specification of a Cauchy problem satisfying these constraints. Therefore we face the question of introducing an external temporal dependence in the quantum dynamics of gravity, by addressing, in close analogy to the approach discussed in Section 2, the kinematical action (5) as till present even in the geometrodynamics framework. Therefore we adopt already on a classical level the full action

$$S^{g\phi k} = \int_{\mathcal{M}^4} \left\{ \pi^{ij}\partial_t h_{ij} + \pi_\phi \partial_t \phi + p_\mu \partial_t y^\mu - N(H^g + H^\phi + p_\mu n^\mu) - N^i(H^g_i + H^\phi_i + p_\mu e^\mu_i) \right\} d^3x dt \tag{23}$$

where n^μ is to be regarded as a functional of $y^\mu(t, x^i)$, but, differently from the case of a fixed background, now h_{ij} (like ϕ) are dynamical variables taken on t and x^i. The variation of this action with respect to the variables h_{jj} π_{ij} ϕ π_ϕ provides the standard Einstein-scalar dynamics, the variation with respect to y^μ and p_μ gives rise respectively to the relations (1) and $\partial_t p_\mu = -p_\nu N \delta n^\nu / \delta y^\mu + \partial_i (N^i p_\mu)$ and finally we have to vary with respect to N and N^i, so getting the constraints

$$H^g + H^\phi = -p_\mu n^\mu \quad H^g_i + H^\phi_i = -p_\mu e^\mu_i \tag{24}$$

However we observe that the equation $\partial_t p_\mu = -p_\nu N \delta n^\nu / \delta y^\mu + \partial_i (N^i p_\mu)$ is a linear and homogeneous first order differential system which ensures that if p_μ is chosen to be zero on the initial hypersurface $\Sigma^3_{t_0}$ then it will remain zero during the whole evolution [1] and the considered theory reduces to ordinary general relativity. Thus we see how, as argued above, the difference in addressing the presence of the kinematical action, respectively to the canonical case relies only on a different structure of the Cauchy problem.

The key point in our reformulation of the canonical quantization of gravity consists of assuming (24) be the appropriate quantum constraints and individualizing the restriction to be imposed on the initial wave functional to get $p_\mu \equiv 0$ when constructing its semiclassical expansion.

In agreement with these considerations we require that the quantum states of the system be represented by a wave functional $\Psi = \Psi(y^\mu,\ h_{ij},\ \phi)$ which, once performed the canonical variables translation into the corresponding operators, satisfy the super-hamiltonian and super-momentum constraints

$$i\hbar n^\mu \frac{\delta \Psi}{\delta y^\mu} = (\hat{H}^g + \hat{H}^\phi_i)\Psi \quad i\hbar e^\mu_i \frac{\delta \Psi}{\delta y^\mu} = (\hat{H}^g_i + \hat{H}^\phi_i)\Psi \tag{25}$$

The second of the above equations exprimes the non-invariance of the wave functional under space diffeomorphisms (i. e. $x^{i\prime} = x^{i\prime}(x^i)$) and therefore, since we expect the quantization procedure, even in a (3+1)-representation of space-time, should yet preserve the tensorial nature of the 3-geometry, then it is natural to require the left-hand-side of of this equation vanish (on a classical level it corresponds to the restriction $p_\mu \propto n_\mu \Rightarrow p_\mu e^\mu_i = 0$); thus we take, as describing the quantum geometrodynamics, the following system of functional differential equations:

$$i\hbar n^\mu \frac{\delta \Psi}{\delta y^\mu} = (\hat{H}^g + \hat{H}^\phi_i)\Psi \quad (\hat{H}^g_i + \hat{H}^\phi_i)\Psi = 0 \quad \Psi = \Psi(\{h_{ij}\}, y^\mu) \tag{26}$$

where now the wave functional is taken again on the 3-geometries ($\{h_{ij}\}$) related by the 3-diffeomorphisms.
These $(4 \times \infty^3)$ equations, which correspond to a natural extension of the Wheeler-DeWitt approach, have the fundamental feature that the first of them is now a parabolic

[1] We remark that if p_μ vanishes on the initial hypersurface, then this equation implies that all its initial time derivatives vanish.

one and it is just this their property which allows to overcome some of the above discussed limits of the WDE.

Though this set of equations provides a satisfactory description of the 3-geometries quantum dynamics, nevertheless it turns out convenient and physically meaningful to take, by (1), the wave functional evolution along a one-parameter family of spatial hypersurfaces filling the universe.

By (1) the firsts of equations (26) can be rewritten as follow (taking into account even the second one of (26)):

$$i\hbar\frac{\delta\Psi}{\delta y^{\mu}}\partial_t y^{\mu} = N(\hat{H}^g + \hat{H}^{\phi})\Psi \tag{27}$$

Now this set of equations can be (heuristically) rewritten as a single one by integrating over the hypersurfaces Σ_t^3, i. e.

$$i\hbar\partial_t\Psi = i\hbar\int_{\Sigma_t^3}\left\{\frac{\delta\Psi}{\delta y^{\mu}}\partial_t y^{\mu}\right\}d^3x = \hat{\mathcal{H}}\Psi \equiv \left[\int_{\Sigma_t^3} N(\hat{H}^g + \hat{H}^{\phi})d^3x\right]\Psi \tag{28}$$

The above equations (28) and (26) show how in the present approach the wave functional is no longer invariant under infinitesimal displacements of the time variable.

Let us now show that the operator $\hat{\mathcal{H}}$ is an hermitian one, i. e., in the bra-ket Dirac notation, it verifies the relations

$$\langle\Psi_1 \mid \hat{\mathcal{H}}\Psi_2\rangle = \langle\hat{\mathcal{H}}\Psi_1 \mid \Psi_2\rangle \tag{29}$$

being Ψ_1 and Ψ_2 two generic solutions of equations (28) and (26).

We start by choosing in $\hat{H}^g$ the following normal ordering for its kinetic part

$$G_{ijkl}\pi^{ij}\pi^{kl} \rightarrow -\hbar^2\frac{\delta}{\delta h_{ij}}\left(G_{ijkl}\frac{\delta(\ )}{\delta h_{kl}}\right) \tag{30}$$

Hence we have

$$\langle\Psi_1 \mid \hat{\mathcal{H}}\Psi_2\rangle \equiv \int_{\mathcal{F}_t} DhD\phi\int_{\Sigma_t^3} d^3x\Psi_1^* N(\hat{H}^g + \hat{H}^{\phi})\Psi_2 \tag{31}$$

where $\mathcal{F}_t$ denotes the functional space $(\{h_{ij}\}, \phi)$, as referred to the hypersurface Σ_t^3, $DhD\phi$ the associated Lebesgue measure and Ψ^* the complex conjugate wave functional, which satisfies the hermitian conjugate equation of (28) and (26) ones.

By observing that, as shown in [13] and easily checkable by repeating the analysis below performed for the gravitational terms, $\hat{H}^{\phi}$ is an hermitian operator, then it remains to be analyzed the term (the functional integrals on ϕ and h commute with each other and both commute with the space integral) [2]:

$$\int_{\mathcal{F}_t} Dh\left\{\int_{\Sigma_t^3}\Psi_1^*\left[-\frac{16\pi G\hbar^2}{c^3}\frac{\delta}{\delta h_{ij}}\left(G_{ijkl}\frac{\delta\Psi_2}{\delta h_{kl}}\right) - \frac{\sqrt{h}c^{33}R}{16\pi G}\Psi_2\right]\right\} \tag{32}$$

[2] The analysis of this section and of the following one has an heuristic character due to its functional approach and it should be made rigorous by an appropriate discretization on a lattice.

Since $G_{ijkl} = G_{klij}$, it is easy to check the relation

$$\int_{\Sigma_t^3} d^3x \left[\Psi_1^* \frac{\delta}{\delta h_{ij}} \left(G_{ijkl} \frac{\delta \Psi_2}{\delta h_{kl}}\right)\right] = \tag{33}$$

$$\int_{\Sigma_t^3} d^3x \left[\frac{\delta}{\delta h_{ij}} \left(\Psi_1^* G_{ijkl} \frac{\delta \Psi_2}{\delta h_{kl}} - \Psi_2 G_{ijkl} \frac{\delta \Psi_1^*}{\delta h_{kl}}\right) + \frac{\delta}{\delta h_{ij}} \left(G_{ijkl} \frac{\delta \Psi_1^*}{\delta h_{kl}}\right) \Psi_2\right] \tag{34}$$

and then, assuming (as done for the matter field ϕ [13]) the validity in the functional space ($\{h_{ij}\}$) of the functional Gauss theorem (the wave functional is required to vanish on the "boundary' of this space):

$$\int_{\mathcal{F}_t} Dh \int_{\Sigma_t^3} d^3x \frac{\delta}{\delta h_{ij}} (...) = 0 \tag{35}$$

we conclude the proof that $\hat{\mathcal{H}}$ is an hermitian operator.

On the base of this results, by (28) we get

$$\partial_t \langle \Psi_1 \mid \Psi_2 \rangle \equiv \int_{\Sigma_t^3} d^3x \partial_t y^\mu \frac{\delta}{\delta y^\mu} \langle \Psi_1 \mid \Psi_2 \rangle = \tag{36}$$

$$\langle \partial_t \Psi_1 \mid \Psi_2 \rangle + \langle \Psi_1 \mid \partial_t \Psi_2 \rangle = \frac{i}{\hbar} (\langle \hat{\mathcal{H}} \Psi_1 \mid \Psi_2 \rangle - \langle \Psi_1 \mid \hat{\mathcal{H}} \Psi_2 \rangle) = 0 \tag{37}$$

and the generic character of the deformation vector allows us to write the fundamental conservation law

$$\frac{\delta \langle \Psi_1 \mid \Psi_2 \rangle}{\delta y^\mu} = 0 \tag{38}$$

Thus we defined an inner product which turns the space of the solutions to equations (28) and (26) into an Hilbert space (so removing the shortcoming i) of the WDE). Indeed now it is possible to define the notion of a conserved functional probability distribution as:

$$\varrho(y^\mu, \{h_{ij}\}, \phi) \equiv \Psi^* \Psi \quad \langle \Psi \mid \Psi \rangle = 1 \quad \frac{\delta \langle Psi \mid \Psi \rangle}{\delta y^\mu} = 0 \tag{39}$$

Let us now reformulate the dynamics described by the equations (28) and (26) by means of the eigenvalues problems for the equations (26) To this end we expand Ψ in the following functional representation:

$$\Psi(y^\mu, \{h_{ij}\}, \phi) = \int_{^*\mathcal{Y}_t} D\omega \Theta(\omega) \chi_\omega(\{h_{ij}\}, \phi) exp \left\{\frac{i}{\hbar} \int_{\Sigma_t^3} d^3x \int dy^\mu (\omega n_\mu)\right\} \tag{40}$$

where $D\omega$ denotes the Lebesgue measure in the functional space $^*y_t$ of the conjugate function $\omega(x^i)$, Θ a functional valued in this domain and n_μ denotes the covariant normal vector; indeed, once assigned $n^\mu(y^\mu)$, the field n_μ can be written, in general, only formally in a quantum space-time.

Substituting this expansion into equations (26) we get the eigenvalues problems

$$(\hat{H}^g + \hat{H}^\phi)\chi_\omega = \omega\chi_\omega \quad (\hat{H}^g_i + \hat{H}^\phi_i)\chi_\omega = 0 \tag{41}$$

Here $\omega(x^i)$ is not a 3-scalar, but it transforms, under 3-diffeomorphisms, like $\hat{H}^g$ or $\hat{H}^\phi$, so ensuring that ωd^3x, as it should, be an invariant quantity.
Now we observe that, by (1), equation (40) rewrites

$$\Psi(y^\mu, \{h_{ij}\}, \phi) = \int_{^*\mathcal{Y}_t} D\omega\Theta(\omega)\chi_\omega(\{h_{ij}\}, \phi)exp\left\{\frac{i}{\hbar}\int_{\Sigma^3_t} d^3x \int_{t_0}^t dt'\partial_{t'}y^\mu(\omega n_\mu)\right\} = \tag{42}$$

$$\Psi(\{h_{ij}\}, \phi, t) = \int_{^*\mathcal{Y}_t} D\omega\Theta(\omega)\chi_\omega(\{h_{ij}\}, \phi)exp\left\{-\frac{i}{\hbar}\int_{t_0}^t dt' \int_{\Sigma^3_t} d^3x(N\omega)\right\} \tag{43}$$

being t_0 an assigned initial "instant.'
To the same result we could arrive by choosing, without any loss of generality, the coordinates system (t, x^i), i. e. $y^0 \equiv t$, $y^i \equiv x^i$; indeed, for this system, the spatial hypersurfaces have equation $t = const$, i. e. $dy^\mu \rightarrow (dt, 0, 0, 0)$ and we have $n_0 = N$. By other words the wave functional (43) is to be interpreted directly in terms of the time variable t, i. e. $\Psi(\{h_{ij}\}, \phi, t)$ and, in fact, it turns out solution of the wave equation

$$i\hbar\partial_t\Psi(\{h_{ij}\}, \phi, t) = \hat{\mathcal{H}}\Psi(\{h_{ij}\}, \phi, t) \tag{44}$$

The expansion (43) of the wave functional and the eigenvalues problems (41) completely describe the quantum dynamics of the 3-geometries.

Now to conclude our analysis, it needs to

recognize which restriction should be imposed on the initial wave functional, say $\Psi(\{h_{ij}\}, \phi\}, t_0) = \Psi_0(\{h_{ij}\}, \phi)$ $(t = t_0$ defining the initial hypersurface) to get in the classical limit the ordinary general relativity, i. e. the quantum counterpart of the condition $p_{\mu 0} = 0$.

To this aim we preliminary observe that, being $\hat{\mathcal{H}}$ an hermitian operator the same remains valid for $\hat{H}^g$ (by identical proof) and the functionals χ_ω are expected to be an orthonormal basis, i. e. $\langle\chi_\omega \mid \chi_{\omega'}\rangle = \Delta(\omega - \omega')$ (Δ denoting the Dirac functional), on which we can expand Ψ_0; in fact we get the functional relation $\Theta(\omega) = \langle\Psi_0 \mid \chi_\omega\rangle$.

As next step we express the wave functional as follows

$$\Psi \equiv \sqrt{\rho}exp\left\{i\frac{\sigma}{\hbar}\right\} \tag{45}$$

being ρ and σ respectively the modulus and the real phase (up to $\hbar$) of Ψ. By substituting (45) into the equation (44) we get for its real part an expression of the form:

$$-\partial_t\sigma = \int_{\Sigma^3_t}\left\{N\hat{H}J\sigma + \left[\frac{N}{\sqrt{\rho}}(\hat{H}^g + \hat{H}^\phi)\right]\sqrt{\rho}\right\}d^3x \tag{46}$$

where by $\hat{H}J$ we denote the Hamilton-Jacobi operator; we stress how, in the above equation, the terms relative to the functional ρ contain, differently from the Hamilton-Jacobi one, $\hbar$, as well as the imaginary part of (44) (which is an evolutive equation for ρ).
Now we observe that once assigned $\sigma_0 \equiv \sigma(\{h_{ij}\}, \phi, t_0)$, equation (46) allows to calculate on $\Sigma^3_{t_0}$ $(\partial_t \sigma) \mid_{t=t_o}$ and, by iteration, all higher order time derivatives; thus we can expand σ in powers of t near enough to the initial hypersurface, i. e. :

$$\sigma = \sum_{n=0}^{\infty} \frac{1}{n!} (\partial_t)^n \sigma \mid_{t=t_0} (t - t_0)^n \tag{47}$$

This expression permits to extend the solution toward the future by reassigning the Cauchy data in $t_0 + \Delta t$, $\Delta t \ll 1$, and then iterating the procedure indefinitely. But if we require that σ_0 satisfies the restriction

$$\hat{H}J\sigma_0 = 0 \tag{48}$$

than we get

$$\partial_t \sigma \mid_{t=t_0} = terms \sim O(\hbar) + ... \quad \Rightarrow \quad \sigma = \sigma_0 + terms \sim O(\hbar)(t - t_0) + ... \tag{49}$$

This result ensures in the semiclassical limit $\hbar \rightarrow 0$ the envolutive nature of the wave functional phase, i. e. $\sigma = \sigma_0$ (the same is not true for the modulus ρ which remains an evolutive variable even in this limit). Thus an initial wave functional whose modulus is a generic one, but whose phase satisfies the Hamilton-Jacobi equation (48) provides a quantum evolution compatible with the classical limit of general relativity.
However we have to note that to retain the invariance under the 3-diffeomorphisms of the quantization procedure prevents, in general, the achievement of a correct limit for the quantum field theory on a fixed background; indeed (26) provides, on a fixed background, the right dynamics (10) only in those reference frames for which $N^i = 0$.

We conclude our reformulation of the quantum geometrodynamics, by emphasizing how, a restriction on the initial wave functional phase σ_0 (as the one required to get the classical limit of general relativity), does not correspond to a real loss of physical degrees of freedom on $\Sigma^3_{t_0}$, since the only meaningful information we can provide on the initial quantum configuration of the system, consists of the functional probability distribution ρ_0 (which, to get the classical limit, should be peaked around a specific solution of the Einstein equations).

5 The Multi-time and Schrödinger Approach

In this section we provide a schematic formulation of the so-called multi-time approach and of its smeared Schr"odinger version, in view of a comparison with the proposal of previous section.

The multi-time formalism is based on the idea that many gravitational degrees of freedom appearing in the classical geometrodynamics have to be not quantized because are not real physical ones; indeed we have to do with $10\times\infty^3$ variables, i. e. the values of the functions $N\,N^i\,h_{ij}$ in each point of the hypersurface Σ^3, but it is well-known that the gravitational field possesses only 4^3 physical degrees of freedom (in fact the gravitational waves have, in each point of the space, only two independent polarizations and satisfy second order equations).
The first step is therefore to extract the real canonical variables by the transformation

$$\left\{h_{ij}\,\pi^{ij}\right\}\to\{\xi^\mu\,\pi_\mu\}\quad\{H_r\,P^r\}\quad\mu=0,1,2,3\,r=1,2\quad, \tag{50}$$

where H_r, P^r are the four real degrees of freedom, while $\xi^\mu\,\pi_\mu$ play the role of embedding variables.

In terms of this new set of canonical variables, the gravity-"matter" action (15) rewrites

$$S^{g\phi}=\int_{\mathcal{M}^4}\left\{\pi_\mu\partial_t\xi^\mu+P^r\partial_tH_r+\pi_\phi\partial_t\phi-N(H^g+H^\phi)-N^i(H_i^g+H_i^\phi)\right\}d^3xdt\quad, \tag{51}$$

where $H^g=H^g(\xi^\mu,\pi_\mu,H_r,P^r)$. and $H_i^g=H_i^g(\xi^\mu,\pi_\mu,H_r,P^r)$.

Now we provide an ADM reduction of the dynamical problem by solving the Hamiltonian constraint for the momenta π_μ

$$\pi_\mu+h_\mu(\xi^\mu,H_r,P^r,\phi,\pi_\phi)=0 \tag{52}$$

Hence the above action takes the reduced form

$$S^{g\phi}=\int_{\mathcal{M}^4}\{P^r\partial_tH_r+\pi_\phi\partial_t\phi-h_\mu\partial_t\xi^\mu\}\,d^3xdt\quad. \tag{53}$$

Finally the lapse function and the shift vector are fixed by the Hamiltonian equations lost with the ADM reduction, as soon as, the functions $\partial_t\xi^\mu$ are assigned. A choice of particular relevance is to set $\partial_t\xi^\mu=\delta_0^\mu$ which leads to

$$S^{g\phi}=\int_{\mathcal{M}^4}\{P^r\partial_tH_r+\pi_\phi\partial_t\phi-h_0\}\,d^3xdt\quad. \tag{54}$$

The canonical quantization of the model follows by replacing all the Poisson brackets with the corresponding commutators; if we assume that the states of the quantum system are represented by a wave functional $\Psi=\Psi(\xi^\mu,H_r,\phi)$, then the evolution is described by the equations

$$i\hbar\frac{\delta\Psi}{\delta\xi^\mu}=\hat{h}_\mu\Psi\quad, \tag{55}$$

where $\hat{h}_\mu$ are the operator version of the classical Hamiltonian densities.
In its smeared formulation the multi-time approach reduces to the following Schrödinger equation

$$i\hbar\partial_t\Psi = \hat{\langle}\Psi \quad \Psi = \Psi(t, H_r, \phi) \quad . \tag{56}$$

Here $\hat{\langle}$ denote the quantum correspondence to the smeared hamiltonian

$$\langle = \int_{\mathcal{M}^4} \{h_\mu \partial_t \xi^\mu\} \, d^3x dt \quad . \tag{57}$$

Now, observing that the first of equations (26) can be rewritten as follows

$$i\hbar\frac{\delta\Psi}{\delta y^\mu} = -n_\mu(\hat{H}^g + \hat{H}_i^\phi)\Psi \quad , \tag{58}$$

it exists a correspondence between the above multi-time approach and our proposal, viewed by identifying the formulas (23)-(54), (58)-(55) and (28)-(56). But the following two key differences appear evident: i) the embedding variables y^μ are added by hand, while the corresponding ones ξ^μ come from non-physical degrees of freedom; ii) the hamiltonians $\mathcal{H}$ and $\langle$ (as well as their corresponding densities) describe very different dynamical situations.

We show explicitly the parallel between these two approaches by their implementation in a minisuperspace model: a Bianchi type IX Universe containing a self-interacting scalar field. By using Misner variables $(\alpha, \beta_+, \beta_-)$ [3] the classical action describing this system reads:

$$S = \int \left\{ p_\alpha\dot{\alpha} + p_{\beta_+}\dot{\beta}_+ + p_{\beta_-}\dot{\beta}_- + p_\phi\dot{\phi} - cNe^{-3\alpha}\left(-p_\alpha^2 + p_{\beta_+}^2 + p_{\beta_-}^2 + p_\phi^2 + V(\alpha, \beta_\pm, \phi)\right)\right\} dt \quad c = const. \quad , \tag{59}$$

where $\dot{(\,)} \equiv d(\,)/dt$ and the precise form of the potential term V is not relevant for our discussion.

For this model, since the Hamiltonian density is independent of the spatial coordinates, then the multi-time approach and its smeared Schrödinger version overlap, the same being true in our formalism.

In the spirit of our proposal the quantum dynamic of this model is described by the equation

$$i\hbar\partial_t\Psi = cNe^{-3\alpha}\hbar^2\left\{\partial_\alpha^2 - \partial_{\beta_+}^2 - \partial_{\beta_-}^2 - \partial_\phi^2 + V\right\}\Psi \quad \Psi = \Psi(t, \alpha, \beta_\pm, \phi) \quad , \tag{60}$$

to which it should be added the restriction that the initial wave function phase $\sigma_0 = \sigma_0(\alpha, \beta_\pm, \phi)$ satisfies the Hamilton-Jacobi equation

$$\left\{-(\partial_\alpha)^2 + (\partial_{\beta_+})^2 + (\partial_{\beta_-})^2 + (\partial_\phi)^2\right\}\sigma_0 + V = 0 \quad . \tag{61}$$

In this scheme $N(t)$ is an arbitrary function of the label time to be specified when fixing a reference.

To set up the multi-time approach we have to preliminarily perform an ADM reduction of the dynamics (59). By solving the Hamiltonian constraint obtained varying N, we find the relation

$$-p_\alpha \equiv h_{ADM} = \sqrt{p^2_{\beta_+} + p^2_{\beta_-} + p^2_\phi + V} \quad . \tag{62}$$

Therefore action (59) rewrites as

$$S = \int \left\{ p_{\beta_+}\dot{\beta}_+ + p_{\beta_-}\dot{\beta}_- + p_\phi\dot{\phi} - \dot{\alpha}h_{ADM} \right\} dt \quad , \tag{63}$$

Thus we see how α plies the role of an embedding variable (indeed it is related to the Universe volume), while $\beta_\pm$ are the real gravitational degrees of freedom (they describe the Universe anisotropy).
By one of the Hamiltonian equation lost in the ADM reduction (i. e. when varying p_α in (59)), we get

$$\dot{\alpha} = -2cNe^{-3\alpha}p_\alpha = 2cNe^{-3\alpha}h_{ADM} \quad . \tag{64}$$

Hence by setting $\dot{\alpha} = 1$, we fix the lapse function as

$$N = \frac{e^{3\alpha}}{2ch_{ADM}} \quad . \tag{65}$$

The quantum dynamics in the multi-time approach is summarized by the equation

$$i\hbar\partial_\alpha\Psi = \sqrt{-\hbar^2(\partial^2_{\beta_+} + \partial^2_{\beta_-} + \partial^2_\phi) + V}\Psi \qquad \Psi = \Psi(\alpha, \beta_\pm, \phi) \quad . \tag{66}$$

We stress that in this multi-time approach the variable α, i. e. the volume of the Universe, behaves as a "time"-coordinate and therefore the quantum dynamics can not avoid the Universe reaches the cosmological singularity ($\alpha \to -\infty$). On the other hand, in the formalism we proposed, α is on the same footing of the other variables and are admissible "stationary states" for which it is distributed in probabilistic way.
This feature reflects a more general and fundamental difference existing between the two approaches: the multi-time formalism violates the geometrical nature of the gravitational field in view of real physical degrees of freedom, while the proposed quantum dynamics implements this idea only up to the lapse function and the shift vector, but preserves the geometrical origin of the 3-metric field.

6 Physical Interpretation of the Model

A fundamental question we have to answer is about the physical meaning of the kinematical variables adopted in the present reformulation of the quantum dynamics. Indeed in the multi-time approach the corresponding embedding variables have no physical meaning since they do not represent any physical degree of freedom, but simply equivalent ways to represent the same real dynamics in the space-time.
However, in our case, these variables have been added by hand to the geometrodynamics and provide, in general, a violation of the classical symmetries; more precisely, by restricting, as in (48), the initial phase of the wave functional, we violate the symmetry

of general relativity only on a quantum level and they are restored on a classical limit, when $\hbar \rightarrow 0$.

A viable point of view is to regard this feature of the dynamics as a prescription of the nature; by other words we may argue that the only experimental knowledge at our disposition concerns the classical dynamics only (which therefore is to be restored anyway), but noting we know, in principle, about the quantum behavior of gravity (which, in the proposed paradigm we prescribe to be summarized by the present equations).

Another, more physical, point of view relies on observing that any dynamical term able to deform the dynamics of an assigned system with different issues for its evolution, should have a precise physical meaning and can be identified with some kind of field. In what follows, we address this last way of thinking and argue the kinematical term can be interpreted as a matter fluid, in close analogy with the so-called *Gaussian reference fluid* which was first proposed by K. Kuchar and his collaborators, see [20, 14]. In this spirit the restriction (48) is no longer a key requirement for the consistency of the theory, but the validity of the idea requires that this material component be experimentally detected.

We start by schematically reviewing the Gaussian reference approach. In a system of Gaussian coordinates $\{T\,, X^i\}$ $(i = 1, 2, 3)$, the line element of a generic gravitational field takes the form

$$ds^2 = -dT^2 + h_{ij}(T, X^k)dX^i dX^j \quad , \tag{67}$$

and therefore such a system is obtained from a generic one by imposing the covariant Gaussian constraints

$$g^{\mu\nu}\partial_\mu T \partial_\nu T = -1 \quad g^{\mu\nu}\partial_\mu T \partial_\nu X^i = 0 \quad . \tag{68}$$

In the above formula we adopted the notation $\partial_\mu \equiv \partial\ /\partial y^\mu$, being y^μ generic coordinates to which it corresponds the metric tensor $g_{\mu\nu}$.

The idea now consists into add this constraint to the gravity-"matter" action (i. e. into requiring the latter be restricted to Gaussian references) via four Lagrangian multipliers M and M^i; thus we consider an additional action term of the form

$$S^{(G)} = \int_{\mathcal{M}^4} \left\{ M(g^{\mu\nu}\partial_\mu T \partial_\nu T - 1) + M^i (g^{\mu\nu}\partial_\mu T \partial_\nu X^i) \right\} d^3x dt \quad . \tag{69}$$

A careful analysis of the whole variational problem, leads [20] to dynamical constraints of the form

$$p_T + h_T = 0 \quad p_{X^i} + h_{x^i} = 0 \quad , \tag{70}$$

where p_T and p_{X^i} denote the conjugate momenta respectively of the variables T and X^i, while h_T and h_{X^i} correspond to linear combinations of the super-hamiltonian and the super-momentum with coefficients depending on T, X^i and the 3-metric h_{ij}. The linearity of the above constraints with respect to the added momenta, allows to perform a satisfactory quantization of the model in the spirit of a Schrödinger-like formalism. The physical interpretation of this proposal relies on regarding the added term as a

kind of fluid which interact with gravity. Though this point of view is affected by some shortcomings (indeed this fluid has a quite peculiar dynamics), nevertheless it provides an appropriate notion of reference fluid clock.

The link of our formalism with the above Gaussian reference fluid consists of the observation that in Gaussian coordinates the spatial Hypersurfaces have equation $T(y^\mu) = const.$ and therefore the quantity $\partial_\mu T$ describes the covariant normal field n_μ, while the gradients $\partial_\mu X^i$ are provided via the reciprocal vectors e^i_μ as $\partial_\mu X^i = e^i_\mu + (N^i/N)n_\mu$. Hence it is natural to argue that our kinematical term corresponds to a generic reference fluid, having an associated time-like vector n^μ, for which the first of (68) is generalized to the normalization condition $g^{\mu\nu}n_\mu n_\nu = -1$; the second condition (68) does not hold for a generic fluid since we have $g^{\mu\nu}n_\mu\partial_\mu X^i = n^\mu[e^i_\mu + (N^i/N)n_\mu] = -N^i/N \neq 0$. Our parallel with the Gaussian reference fluid becomes precise by stressing how, in our case, the normalization condition for n^μ should not be added as a constraint (by some Lagrangian multiplier); indeed it is ensured by the relation (1), obtained when varying the kinematical action with respect to p_μ.
We recall that the dynamics of the *generic fluid reference* is described by the Hamiltonian equations

$$\partial_t y^\mu = Nn^\mu + N^i\partial_i y^\mu \quad \partial_t p_\mu = -p_\nu N\delta n^\nu/\delta y^\mu + \partial_i(N^i p_\mu) \quad . \tag{71}$$

Once assigned the vector n^μ as a functional of y^μ and the functions $N(t,x^i)$ $N^i(t,x^i)$, we can solve the first of these equations to get the kinematical variables $y^\mu = y^\mu(t,x^i)$; Hence by substituting this information into the second Hamiltonian equation and solving it, we get the momentum $p_\mu = p_\mu(t,x^i)$.

In order to outline the relation existing between this system of Hamiltonian equations and the dynamics of a fluid, we multiply the second of (71) by n^μ, so getting via the first one

$$n^\mu[\partial_t p_\mu + p_\nu N\delta n^\nu/\delta y^\mu - \partial_i(N^i p_\mu)] = \partial_t(p_\mu n^\mu) - \partial_i(N^i p_\mu n^\mu) = 0 \quad . \tag{72}$$

Hence by setting $p_\mu n^\mu = -\bar{\omega}$ and labeling with barred indices $\bar{\mu}, \bar{\nu}, ...$ all the quantities in the coordinates t, x^i, we rewrite the above equation as follows

$$\partial_t(p_\mu n^\mu) - \partial_i(N^i p_\mu n^\mu) = -\partial_{\bar{\mu}}(N\bar{\omega}n^{\bar{\mu}}) = 0 \;\Rightarrow \nabla_{\bar{\mu}}(\varepsilon n^{\bar{\mu}}) = 0 \quad , \tag{73}$$

where $\varepsilon \equiv \bar{\omega}/\sqrt{h}$ denotes a real 3-scalar and ∇ refers to the covariant 4-derivative. By the covariance of this equation, we should have in general

$$\nabla_\mu(\varepsilon n^\mu) = -n^\mu\nabla_\nu t^\nu_\mu = 0 \quad , \tag{74}$$

being $t^\nu_\mu \equiv \varepsilon n_\mu n^\nu$ the energy-momentum tensor of a dust with energy density ε and four-velocity $n^\mu = n^\mu(t,x^i)$. The real correspondence of this tensor with the kinematical term, i. e. with the *generic fluid reference*, comes out when observing, first that the requirement $e^\mu_i\delta\Psi/\delta y^\mu = 0$ implies, on a classical level $p_\mu e^\mu_i = 0$ and then how the kinematical term $-p_\mu n^\mu$, appearing on the right-hand-side of the super-Hamiltonian constraint can be rewritten in the expressive form

$$- p_\mu n^\mu = \bar{\omega} = \sqrt{h}\varepsilon = \sqrt{h} t_{\mu\nu} n^\mu n^\nu = \frac{1}{N^2}\sqrt{h} t_{\mu\nu}\partial_t y^\mu \partial_t y^\nu = \frac{1}{N^2} t_{\bar{0}\bar{0}} = -\sqrt{h} t^{\bar{0}}_{\bar{0}} \tag{75}$$

(of course, apart from $t_{\bar{0}\bar{0}}$, all the other components $t_{\bar{\mu}\bar{\nu}}$ vanish identically).

Therefore the super-Hamiltonian constraint acquires the familiar expression

$$H^g + H^\phi - \bar{\omega} = H^g + H^\phi + \sqrt{h} t^{\bar{0}}_{\bar{0}} = 0 \quad . \tag{76}$$

Thus we see how the *generic reference fluid*, having the energy-momentum tensor of a dust fluid, contributes to the full super-Hamiltonian by a quantity like the one due to a space-time dependent cosmological term, coinciding with its energy density.

In the light of these considerations, the requirement that p_μ vanishes identically implies that $\varepsilon \equiv 0$ and therefore such a reference fluid does not interact with gravity by its "energy-momentum"; by a purely phenomenological point of view, in this case, it behaves like a test "matter" field, whose kinematics is fixed by (1).

This identification with a reference fluid allows us to achieve the fundamental result of upgrading the formal time y^μ to a real physical clock.

Finally we stress the following two points:

i) The momentum equation is equivalent to the conservation law for the dust energy-momentum tensor and therefore provides the behavior of ε. For a Gaussian system this equation yields $\varepsilon = c(x^i)/\sqrt{h} \;\;\Rightarrow\;\; \bar{\omega} \equiv c(x^i)$, being c a generic space function; This behavior is just the one of a dust energy density, but it is worth noting that our analysis does not oblige ε to be a positively defined quantity.

ii) The classical limit of the super-Hamiltonian equation (26) provides an Hamilton-Jacobi equation where the eigenvalue ω coincides (see (76) with the function $\bar{\omega}$ taken on a specific hypersurface.

7 Concluding Remarks and a Simple model

As outcoming of our analysis, we get a reformulation of the canonical quantization of gravity in which is removed the so-called "frozen formalism" typical of the WDE, i. e. the wave functional becomes evolutive along a one-parameter family of spatial hypersurfaces filling the space-time and can be expressed in terms of a direct dependence on the parameter t labeling the slicing (this result provides a solution to the shortcoming of the WDE emphasized at the point ii)); indeed the existence of an Hilbert space associated to the solutions of the restated equation can be regarded as a consequence of the non-frozen formalism here obtained (resulting into a parabolic nature of the super-hamiltonian quantum constraint).

Instead of these successful results, the WDE shortcoming indicated at the point iii) is overcome only with respect to those reference frames where the shift vector vanishes; the reasons for this incompleteness are to be regarded as a direct consequence of retaining in the quantum geometrodynamics the invariance of the wave functional under the 3-diffeomorphisms.

Of course the relaxation of this restriction is a possible issue of an extended theory; here we do not address this point because it implies some non-trivial complications in constructing an Hilbert space (at least on our heuristic level), but overall because it would correspond to leave *a priori* the notion of a covariant 3-geometrodynamics, without any strong physical motivation to pursue this way (indeed we are forced to relax the time displacement invariance by the incompatibility of a quantum space-time and the (3 + 1)-slicing notion).

However the main goal of our analysis is achieved by removing the shortcoming of the WDE stated at the point iv), i. e. now the quantization procedure takes place in a fixed system (indeed only the lapse function should be specified, while the shift vector can assume a generic value) and no ambiguity survives about the time-like character of the normal field; by other words, in this new approach it is possible to quantize the 3-geometry field on a fixed family of spatial hypersurfaces (corresponding to its evolution in the space-time), because this quantization scheme does not contradicts the strong assumption of a (3 + 1)-slicing of the 4-dimensional manifold.

The discussion presented in Section 6, about the interpretation of the kinematical variables as a generic reference fluid clock, has the very important merit to transform a working formalism into a possible experimental issue; however it should be supported by further investigation on the physical consequences the clock fluid has when referred to specific contexts: indeed the question about the appropriate definition of a reference frame in a quantum space-time is a really subtle one (see for instance [22, 23]);

In spite of this available physical issue (ensured by the fluid interpretation), we emphasize how to have found the (non-physical) restriction on the initial wave functional phase (48), which ensures the classical limit coinciding with general relativity, is essential for the consistency of the whole approach; in fact the physical meaning of this result concerns the fundamental achievement of restoring, on a classical level, the invariance of the theory under the time displacements, which is instead broken by the quantum dynamics.

We conclude by observing how the functional nature of all our approach implies it has (like in the WDE) a heuristic value; but it appear rather reasonable that it can be made rigorous when reformulated, in a discrete approach, as a theory on a suitable "lattice".

Though is out of the aim of this paper to face this problem, (to be regarded as a fundamental subject of further investigations), nevertheless we here suggest that the best method to reformulate the quantum dynamics on a discrete level, seems to be via the Regge calculus [24, 25] as applied to the 3-geometry field. The physical justification for a discrete approach to quantum gravity relies on the expectation that the space-time has a *lattice structure* (or a granular morphology) on a Planckian scale.

At the end of this work we provide an application of the obtained theory to the quantization of a very simple model, described by the following line element:

$$ds^2 = N(t)^2 dt^2 - r(t)^{4/3}\delta_{ij}dx^i dx^j \quad , \tag{77}$$

where δ_{ij} denote the Krönecker matrix and the flat hypersurfaces $t = const.$ are

taken to have a closed topology, i. e. $0 \leq x^i < 2\pi L$ $(i = 1, 2, 3)$ ($L = const.$ is the radius of the three cyrcles and has the dimensionality of a length). We allow r belongs to the positive real axis $0 \leq r < \infty$ (since the metric is invariant under the exchange $r \rightarrow -r$.

The ADM action describing the vacuum dynamics of this "1-dimensional" model, reads in the simple form

$$S = -\frac{\pi^2 L^3 c^3}{2G} \int \dot{r}^2 dt = \int \left\{ p_r \dot{r} - \frac{GN}{2\pi^2 L^3 c^3} (-p_r^2) \right\} dt \quad , \tag{78}$$

where $\dot{(\,)} \equiv \partial_t(\,)$ and p_r denotes the conjugate momentum to the variable r. From a classical point of view this model corresponds to the Euclidean 3-space (to which is associated a generic time variable) since $\dot{r} \propto p_r = 0 \Rightarrow r = r_0 = const.$ and the corresponding Hamilton-Jacobi equation and solution read

$$\left(\frac{d\sigma}{dr} \right)^2 = 0 \Rightarrow \sigma = \sigma_0 = const, \quad . \tag{79}$$

The quantum dynamics of the model is described by equation (28), which, in the present case reduces to the very simple form:

$$i\hbar \partial_t \Psi(t, r) = \frac{N\hbar^2}{2\mu} \partial_r^2 \Psi(t, r) \quad \mu \equiv \frac{\pi^2 L^3 c^3}{G} \quad . \tag{80}$$

Apart from the negative nature of its hamiltonian, we see how this quantum gravity model corresponds to the free nonrelativistic particle; the general solution of the above equation reads in the following wave-packet form:

$$\Psi(t, r) = \frac{1}{\sqrt{\hbar}} \int_{-\infty}^{\infty} dp \varphi(p) exp \left\{ \frac{i}{\hbar} \left(pr + \frac{p^2}{2\mu} \int_{t_0}^{t} N(t') dt' \right) \right\} \quad , \tag{81}$$

where $\varphi(p)$ is determined by the relation

$$\varphi(p) = \frac{1}{\sqrt{\hbar}} \int_0^{\infty} dr \Psi_0(r) exp \left\{ -i \frac{pr}{\hbar} \right\} \quad \Psi_0(r) \equiv \Psi(t = 0, r) \quad . \tag{82}$$

Since Ψ (as well as Ψ_0) should verify the boundary conditions $\Psi(r = 0) = 0$ (in addition to $\Psi(r \rightarrow \infty) = 0$), then φ should be antisymmetric in its argument, i. e. $\varphi(-p) = -\varphi(p)$.

In terms of the expression (82) and by choosing the synchronous gauge $N = c$ $(t \rightarrow T)$, we may rewrite the wave function (81) as follows (taking into account the expression for μ and setting $T_0 = 0$):

$$\Psi(T, r) = e^{i\sigma_0/\hbar} \frac{1}{2\pi} \int_0^{\infty} dr' \chi_0(r') \int_{-\infty}^{\infty} dx exp \left\{ i \left[x(r - r') + x^2 \frac{cT {l_{Pl}}^2}{2\pi^2 L^3} \right] \right\} \quad , \tag{83}$$

where l_{Pl} denotes the Planck length ($l_{Pl} \equiv \sqrt{G\hbar/c^3}$) and, in agreement with the restriction (48), we set

$$\Psi_0 = \chi_0 e^{i\sigma_0/\hbar} \quad , \tag{84}$$

being χ_0 a real function subjected only to the boundary conditions $\chi_0(r = 0) = 0$ and $\chi_0(r \to \infty) = 0$.

At last, it is worth noting that, for any fixed T, takes place the (intriguing) limit

$$\lim_{(cT{l_{Pl}}^2/L^3)\to 0} \Psi(T, r) = \Psi_0(r) \quad , \tag{85}$$

where the initial probability distribution χ_0^2 can eventually be regarded as strongly peaked around a fixed value $r = r_0$.

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